hep-th/9710068

IPM-97-241

October 1997

Periods and Prepotential in $N = 2$ Supersymmetric E_6 Yang-Mills Theory

A.M. Ghezelbash[1]

Institute for Studies in Theoretical Physics and Mathematics,

P.O. Box 19395-5531, Tehran, Iran.

Department of Physics, Alzahra University, Vanak, Tehran 19834, Iran.

Abstract

We obtain the periods and one-instanton coefficient of the $N = 2$ supersymmetric Yang-Mills theory with the exceptional gauge group E_6. These calculations are based on the E_6 spectral curve and the obtained one-instanton coefficient is in agreement with the microscopic results.

[1] e-mail:amasoud@physics.ipm.ac.ir

In the last few years, enormous advances have been made in understanding of the low-energy behaviours of $N = 2$ supersymmetric gauge theories. Progress began with the paper of Seiberg and Witten [1], where the exact low-energy Wilsonian effective action for the pure $N = 2$ supersymmetric Yang-Mills theory with the gauge group $SU(2)$ is derived. Since then, their work has been generalized to other pure gauge groups [2] and to theories with the matter multiplet [3]. In principle, the exact solution of such theories is given by an algebraic curve. In the case of theories with classical Lie gauge groups, algebraic curves are hyperelliptic [4] which must be satisfied in the consistency conditions of the relevant theories. The hyperelliptic curve of the theories with the exceptional gauge groups are constructed in [5, 6].

To understand the strong coupling region of the theory, one considers the vevs of the Higgs fields $\vec{a}$ and their duals $\vec{a}_D$ which are related to the prepotential of the low-energy effective action, $F(\vec{a})$, by $\vec{a}_D = \frac{\partial F}{\partial \vec{a}}$. These fields which are periods of the Riemann surface $\pi = \begin{pmatrix} \vec{a}_D \\ \vec{a} \end{pmatrix}$, are represented by the contour integrals of the Seiberg-Witten differential one-form, λ,

$$\vec{a} = \int_{\vec{\alpha}} \lambda, \qquad \vec{a}_D = \int_{\vec{\beta}} \lambda, \tag{1}$$

where $\vec{\alpha}$ and $\vec{\beta}$ are homology cycles on the Riemann surface.

To obtain the periods, $\vec{a}$ and $\vec{a}_D$, the Picard-Fuchs (PF) operators, which annihilate $\vec{a}$ and $\vec{a}_D$ are usually used. The PF equations have been derived in the case of pure classical gauge groups and also classical gauge groups with massless and massive supermultiplets [7]. In all these cases, the underlying algebraic curves of the theory are hyperelliptic curves. On the other hand, the algebraic curve for the supersymmetric gauge theory is constructed from the spectral curve of the periodic Toda lattice [8]. In the case of classical gauge groups, these curves are equivalent to the hyperelliptic curves. The PF equations for the supersymmetric G_2 gauge group has been constructed from the spectral curve of the $(G_2^{(1)})^\vee$ Toda lattice theory [9]. Also it has been shown that the calculation of n-instanton effects agree with the microscopic results [10], while the calculation of one-instanton effect based on the hyperelliptic curve shows different behaviour with the microscopic results.

Our motivation in this and subsequent works is to shed light on the strong behaviour of E_6 theory and compare the spectral and hyperelliptic curves of the theory by finding the form of the periods and the prepotential of the E_6 theory. The PF equations of E_6 gauge theory based on the spectral curve have

been obtained in [11, 12]. In [11], PF equations are obtained from the topological two dimensional Landau-Ginzburg theory, while in [12], the PF equations are constructed from the basic equations of the theory. However, a comparison of the PF equations, shows their equivalence. In this paper, we will give solutions of these equations, and calculate the leading instanton corrections and find that the one-instanton contribution of the prepotential coincides with the one obtained by using the direct instanton calculation [10].

We use the spectral curve of E_6 given by [13],

$$\zeta + \frac{w}{\zeta} = -u_6 + \frac{q_1 + p_1\sqrt{p_2}}{x^3}, \tag{2}$$

which is obtained by degeneration of K_3 surface to an E_6 type singularity. The polynomials q_1,p_1 and p_2 are given by,

$$\begin{aligned}
q_1 &= 270x^{15} + 342u_1x^{13} + 162u_1^2x^{11} - 252u_2x^{10} + (26u_1^3 + 18u_3)x^9 - 162u_1u_2x^8 + (6u_1u_3 - 27u_4)x^7 \\
&- (30u_1^2u_2 - 36u_5)x^6 + (27u_2^2 - 9u_1u_4)x^5 - (3u_2u_3 - 6u_1u_5)x^4 - 3u_1u_2^2x^3 - 3u_2u_5x - u_2^3, \\
p_1 &= 78x^{10} + 60u_1x^8 + 14u_1^2x^6 - 33u_2x^5 + 2u_3x^4 - 5u_1u_2x^3 - u_4x^2 - u_5x - u_2^2, \\
p_2 &= 12x^{10} + 12u_1x^8 + 4u_1^2x^6 - 12u_2x^5 + u_3x^4 - 4u_1u_2x^3 - 2u_4x^2 + 4u_5x + u_2^2,
\end{aligned} \tag{3}$$

where u_1, u_2, u_3, u_4, u_5 and u_6 are the Casimirs of E_6.

We use the PF equations of [11], and by introducing the new variables,

$$x_1 = \frac{u_4u_5}{u_2u_6},\ x_2 = \frac{u_6}{u_1^2u_4},\ x_3 = \frac{u_6}{u_1u_2^2},\ x_4 = \frac{u_1u_3}{u_4},\ x_5 = \frac{u_6}{u_3^2},\ x_6 = \frac{w}{u_6^2}, \tag{4}$$

we get the following differential operators,

$$\begin{aligned}
E_1 &= 32x_5\theta_3(\theta_3 - 1) - 6\theta_{16} + 3\theta_{26} - 6\theta_{36} + 3\theta_{56} - 3\theta_6, \\
E_2 &= 3x_1\theta_{26} - \theta_{45}, \\
E_3 &= 4x_1x_4x_5\theta_{23} - \theta_{15}, \\
E_4 &= 4x_4x_5\theta_{34} - 3\theta_{16}, \\
E_5 &= x_1x_2x_4\theta_{14} + 8x_1x_4x_5\theta_3(\theta_3 - 1) + 2\theta_{35} + 6x_1\theta_{36}, \\
E_6 &= 2x_1x_2x_4^2x_5\theta_4(\theta_4 - 1) + 12x_1x_4x_5\theta_{36} + 3\theta_{56} + 9x_1\theta_6(\theta_6 - 1), \\
E_7 &= x_3\theta_2(\theta_2 - 1) - 12x_2\theta_{14} - 18\theta_{16} - 18\theta_{26} + 3x_2x_4\theta_4(\theta_4 - 1) - 18\theta_{46} - 18\theta_6,
\end{aligned}$$

$$
\begin{aligned}
E_8 &= 4x_1x_2\theta_1(\theta_1-1)+8x_1x_4x_5\theta_{13}-x_1x_2x_4\theta_{14}+\theta_{15}+8x_1x_4x_5\theta_{34}+2x_1^2x_2x_4^2x_5\theta_{24}-\theta_{25}-\theta_{45}\\
&+ 12x_1x_4x_5\theta_3+3x_1\theta_6,\\
E_9 &= 2x_1^2x_2x_4^2x_5\theta_{24}+4x_1x_4x_5\theta_{35}+\theta_5(\theta_5-1)+3x_1\theta_{56},\\
E_{10} &= 8\theta_{13}-2x_4\theta_{14}+x_4\theta_{24}-2x_4\theta_{34}+x_4\theta_{45},\\
E_{11} &= 4\theta_1(\theta_1-1)+20\theta_{12}+24\theta_{13}+32\theta_{14}+36\theta_{15}+48\theta_{16}+25\theta_2(\theta_2-1)+60\theta_{23}+80\theta_{24}+90\theta_{25}+120\theta_{26}\\
&+ 36\theta_3(\theta_3-1)+96\theta_{34}+108\theta_{35}+144\theta_{36}+64\theta_4(\theta_4-1)+144\theta_{45}+192\theta_{46}+162\theta_5(\theta_5-1)+216\theta_{56}\\
&+ 144(1-4x_6)\theta_6(\theta_6-1)+15\theta_2+24\theta_3+48\theta_4+63\theta_5+120\theta_6+1,\\
E_{12} &= x_3\theta_{15}+6x_1x_2x_4^2x_5\theta_4(\theta_4-1)+18x_1^2x_2x_4^2x_5\theta_{16},\\
E_{13} &= 36x_1x_2\theta_{16}-x_3\theta_{25}-12x_1x_2x_4^2x_5\theta_4(\theta_4-1)-9x_1x_2x_4\theta_{46},\\
E_{14} &= 2x_3\theta_{35}+9x_1x_2x_4\theta_{46}+27x_1^2x_2x_4\theta_6(\theta_6-1),\\
E_{15} &= x_3\theta_5(\theta_5-1)-144x_1^2x_2x_4x_5\theta_{36}+36x_1^2x_2x_4^2x_5\theta_{46}+27\theta_6(\theta_6-1),\\
E_{16} &= x_1x_2x_3x_4\theta_{12}-9x_1x_2x_4(\theta_{16}+\theta_{26}+\theta_{56}+\theta_6)-36x_1^2x_2x_4^2x_5\theta_{36}+2x_3\theta_{35}. \qquad (5)
\end{aligned}
$$

In the above equations, θ_i's are the Euler derivatives $\theta_i = u_i\frac{\partial}{\partial u_i}$ which are related to the Euler derivatives $\varphi_i = x_i\frac{\partial}{\partial x_i}$ by the following relations,

$$
\theta_1=-2\varphi_2-\varphi_3+\varphi_4,\theta_2=-\varphi_1-2\varphi_3,\theta_3=\varphi_4-2\varphi_5,\theta_4=\varphi_1-\varphi_2-\varphi_4,\theta_5=\varphi_1,\theta_6=-\varphi_1+\varphi_2+\varphi_3+\varphi_5-2\varphi_6, \qquad (6)
$$

and $\theta_{ij}=\theta_i\theta_j$.

Now, we construct the solutions of the PF equations $E_i\pi=0\,(i=0,\cdots,16)$ in the semi-classical region where w is small. We consider the following power series solution around $(x_1,x_2,x_3,x_4,x_5,x_6)=(0,0,0,0,0,0)$,

$$
\begin{aligned}
\rho_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6}(x_1,x_2,x_3,x_4,x_5,x_6) &= \sum_{m_1,m_2,m_3,m_4,m_5,m_6\geq 0} c_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6}(m_1,m_2,m_3,m_4,m_5,m_6)\\
& x_1^{m_1+\alpha_1}x_2^{m_2+\alpha_2}x_3^{m_3+\alpha_3}x_4^{m_4+\alpha_4}x_5^{m_5+\alpha_5}x_6^{m_6+\alpha_6}, \qquad (7)
\end{aligned}
$$

where $c_{\alpha_1,\cdots,\alpha_6}(0,\cdots,0)=1$. By inserting eq. (7) in the eqs. (5), we get the following indicial equations,

$$
\begin{aligned}
& 4\lambda_1(\lambda_1-1)+20\lambda_1\lambda_2+24\lambda_1\lambda_3+32\lambda_1\lambda_4+36\lambda_1\lambda_5+48\lambda_1\lambda_6+25\lambda_2(\lambda_2-1)+60\lambda_2\lambda_3+80\lambda_2\lambda_4+90\lambda_2\lambda_5\\
+& 120\lambda_2\lambda_6+36\lambda_3(\lambda_3-1)+96\lambda_3\lambda_4+108\lambda_3\lambda_5+144\lambda_3\lambda_6+64\lambda_4(\lambda_4-1)+144\lambda_4\lambda_5+192\lambda_4\lambda_6
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & 162\lambda_5(\lambda_5-1)+216\lambda_5\lambda_6+144\lambda_6(\lambda_6-1)+15\lambda_2+24\lambda_3+48\lambda_4+63\lambda_5+120\lambda_6+1=0, \\
& \lambda_6(2\lambda_1-\lambda_2+2\lambda_3-\lambda_5+1)=0, \lambda_6(\lambda_1+\lambda_2+\lambda_4+1)=0, \lambda_5(\lambda_1-\lambda_2-\lambda_4)=0, \lambda_5(\lambda_5-1)=0, \lambda_4(\lambda_4-1)=0, \\
& \lambda_6(\lambda_6-1)=0, \lambda_4\lambda_5=0,\ \lambda_1\lambda_5=0,\ \lambda_1\lambda_6=0,\ \lambda_3\lambda_5=0,\ \lambda_5\lambda_6=0,\ \lambda_1\lambda_3=0,\ \lambda_4\lambda_6=0, \qquad (8)
\end{aligned}
$$

where

$$
\lambda_1=-2\alpha_2-\alpha_3+\alpha_4, \lambda_2=-\alpha_1-2\alpha_3, \lambda_3=\alpha_4-2\alpha_5, \lambda_4=\alpha_1-\alpha_2-\alpha_4, \lambda_5=\alpha_1, \lambda_6=-\alpha_1+\alpha_2+\alpha_3+\alpha_5-2\alpha_6. \qquad (9)
$$

The equations (8) have the following solutions,

$$
\begin{aligned}
(\alpha_1,\cdots,\alpha_6) &= (0,1/6,-1/6,-1/6,-1/12,-1/24), \\
(\alpha_1,\cdots,\alpha_6) &= (0,-1/6,1/2,1/6,-5/12,-1/24), \\
(\alpha_1,\cdots,\alpha_6) &= (0,\alpha_2,-3\alpha_2,-\alpha_2,2\alpha_2-1/12,-1/24), \\
(\alpha_1,\cdots,\alpha_6) &= (0,\alpha_2,-1/2\alpha_2-1/12,-\alpha_2,-1/2\alpha_2,-1/24), \\
(\alpha_1,\cdots,\alpha_6) &= (0,-2\alpha_3+5/6,\alpha_3,2\alpha_3-11/6,\alpha_3-11/12,-1/24), \\
(\alpha_1,\cdots,\alpha_6) &= (0,\alpha_2,-3\alpha_2-1,-\alpha_2-1,2\alpha_2+11/12,-1/24). \qquad (10)
\end{aligned}
$$

The coefficients $c_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6}(m_1,m_2,m_3,m_4,m_5,m_6)$ obey the recursion relations,

$$
\begin{aligned}
c_{\vec{\alpha}}(\vec{m}) &= A_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1,m_2,m_3,m_4,m_5-1,m_6), \\
c_{\vec{\alpha}}(\vec{m}) &= B_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2,m_3,m_4,m_5,m_6), \\
c_{\vec{\alpha}}(\vec{m}) &= C_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2,m_3,m_4-1,m_5-1,m_6), \\
c_{\vec{\alpha}}(\vec{m}) &= D_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1,m_2,m_3,m_4-1,m_5-1,m_6), \\
c_{\vec{\alpha}}(\vec{m}) &= E_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2-1,m_3,m_4-1,m_5,m_6)+F_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2,m_3,m_4-1,m_5-1,m_6) \\
&+ G_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2,m_3,m_4,m_5,m_6), \\
c_{\vec{\alpha}}(\vec{m}) &= H_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2-1,m_3,m_4-2,m_5-1,m_6)+I_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2,m_3,m_4-1,m_5-1,m_6) \\
&+ J_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2,m_3,m_4,m_5,m_6), \\
c_{\vec{\alpha}}(\vec{m}) &= K_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1,m_2,m_3-1,m_4,m_5,m_6)+L_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1,m_2-1,m_3,m_4,m_5,m_6) \\
&+ M_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1,m_2-1,m_3,m_4-1,m_5,m_6),
\end{aligned}
$$

$$
\begin{aligned}
c_{\vec{\alpha}}(\vec{m}) &= N_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2-1,m_3,m_4,m_5,m_6)+(O_{\vec{\alpha}}+Q_{\vec{\alpha}}+S_{\vec{\alpha}})(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2,m_3,m_4-1,m_5-1,m_6)\\
&+ P_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2-1,m_3,m_4-1,m_5,m_6)+R_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-2,m_2-1,m_3,m_4-2,m_5-1,m_6)\\
&+ T_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2,m_3,m_4,m_5,m_6),\\
c_{\vec{\alpha}}(\vec{m}) &= U_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-2,m_2-1,m_3,m_4-2,m_5-1,m_6)+V_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2,m_3,m_4-1,m_5-1,m_6)\\
&+ W_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1-1,m_2,m_3,m_4,m_5,m_6),\\
c_{\vec{\alpha}}(\vec{m}) &= X_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1,m_2,m_3,m_4-1,m_5,m_6),\\
c_{\vec{\alpha}}(\vec{m}) &= Y_{\vec{\alpha}}(\vec{m})c_{\vec{\alpha}}(m_1,m_2,m_3,m_4,m_5,m_6-1),
\end{aligned}
\tag{11}
$$

where the functions $A_{\vec{\alpha}}(\vec{m}),\cdots,Y_{\vec{\alpha}}(\vec{m})$ are given in the appendix. Therefore from these recursion relations, one can find the coefficients $c_{\vec{\alpha}}(\vec{m})$, which some of them are as follows,

$$
\begin{aligned}
c_{\vec{\alpha}}(m_1,0,0,0,0,0)=\prod_{i=1}^{m_1}B_{\vec{\alpha}}(i,0,0,0,0,0) \quad &, \quad c_{\vec{\alpha}}(0,m_2,0,0,0,0)=\prod_{i=1}^{m_2}L_{\vec{\alpha}}(0,i,0,0,0,0)\\
c_{\vec{\alpha}}(0,0,m_3,0,0,0)=\prod_{i=1}^{m_3}K_{\vec{\alpha}}(0,0,i,0,0,0) \quad &, \quad c_{\vec{\alpha}}(0,0,0,m_4,0,0)=\prod_{i=1}^{m_4}X_{\vec{\alpha}}(0,0,0,i,0,0)\\
c_{\vec{\alpha}}(0,0,0,0,m_5,0)=\prod_{i=1}^{m_5}A_{\vec{\alpha}}(0,0,0,0,i,0) \quad &, \quad c_{\vec{\alpha}}(0,0,0,0,0,m_6)=\prod_{i=1}^{m_6}Y_{\vec{\alpha}}(0,0,0,0,0,i)
\end{aligned}
$$

$$
\begin{aligned}
&c_{\vec{\alpha}}(m_1,m_2,0,0,0,0)=N_{\vec{\alpha}}(m_1-1,m_2-1,0,0,0,0)N_{\vec{\alpha}}(m_1-2,m_2-2,0,0,0,0)\cdots N_{\vec{\alpha}}(\{\mid m_1-m_2\mid,0\},0,0,0,0)\times\\
&c_{\vec{\alpha}}(\{\mid m_1-m_2\mid,0\},0,0,0,0)+\sum_{i=0}^{\min\{\mathrm{m_1,m_2}\}}\prod_{j=1}^{i}N_{\vec{\alpha}}(m_1-j,m_2-j,0,0,0,0)\times\\
&T_{\vec{\alpha}}(m_1-i-1,m_2-i-1,0,0,0,0)c_{\vec{\alpha}}(m_1-i-1,0,0,0,0,0),\\
&c_{\vec{\alpha}}(m_1,0,m_3,0,0,0)=\prod_{i=1}^{m_1}Z_{\vec{\alpha}}(i,0,m_3+1,0,0,0)c_{\vec{\alpha}}(0,0,m_3,,0,0,0),
\end{aligned}
\tag{12}
$$

where if $m_1>m_2$, $\{\mid m_1-m_2\mid,0\}=m_1-m_2,0$ and else $\{\mid m_1-m_2\mid,0\}=0,m_2-m_1$ and similar complicated structure for the other coefficients. The function $Z_{\vec{\alpha}}$ is given in the appendix . To obtain the other solutions of eqs. (5), we apply the well known Frobenius method, and find the logarithmic solutions of (5). Hence the solutions of the PF equations (5) are given by,

$$
\begin{aligned}
\rho^{(1)}_{\vec{a}}(\vec{x}) &= \rho_{(0,1/6,-1/6,-1/6,-1/12,-1/24)}(\vec{x}),\\
\rho^{(2)}_{\vec{a}}(\vec{x}) &= \rho_{(0,-1/6,1/2,1/6,-5/12,-1/24)}(\vec{x}),\\
\rho^{(3)}_{\vec{a}}(\vec{x}) &= \rho_{(0,\alpha_2,-3\alpha_2,-\alpha_2,2\alpha_2-1/12,-1/24)}(\vec{x}),
\end{aligned}
$$

$$
\begin{aligned}
\rho^{(4)}_{\vec{a}}(\vec{x}) &= \rho_{(0,\alpha_2,-1/2\alpha_2-1/12,-\alpha_2,-1/2\alpha_2,-1/24)}(\vec{x}),\\
\rho^{(5)}_{\vec{a}}(\vec{x}) &= \rho_{(0,-2\alpha_3+5/6,\alpha_3,2\alpha_3-11/6,\alpha_3-11/12,-1/24)}(\vec{x}),\\
\rho^{(6)}_{\vec{a}}(\vec{x}) &= \rho_{(0,\alpha_2,-3\alpha_2-1,-\alpha_2-1,2\alpha_2+11/12,-1/24)}(\vec{x}),\\
\rho^{(1)}_{D\ \vec{a}}(\vec{x}) &= (\frac{\partial}{\partial\alpha_2}-\frac{\partial}{\partial\alpha_3}-\frac{\partial}{\partial\alpha_4}+\frac{1}{2}\frac{\partial}{\partial\alpha_5}+\frac{1}{4}\frac{\partial}{\partial\alpha_6})\rho_{\vec{\alpha}}(\vec{x})\mid_{\vec{\alpha}=(0,1/6,-1/6,-1/6,-1/12,-1/24)},\\
\rho^{(2)}_{D\ \vec{a}}(\vec{x}) &= (\frac{\partial}{\partial\alpha_2}-3\frac{\partial}{\partial\alpha_3}-\frac{\partial}{\partial\alpha_4}+\frac{5}{2}\frac{\partial}{\partial\alpha_5}+\frac{1}{4}\frac{\partial}{\partial\alpha_6})\rho_{\vec{\alpha}}(\vec{x})\mid_{\vec{\alpha}=(0,-1/6,1/2,1/6,-5/12,-1/24)},\\
\rho^{(3)}_{D\ \vec{a}}(\vec{x}) &= (\frac{\partial}{\partial\alpha_2}-3\frac{\partial}{\partial\alpha_3}-\frac{\partial}{\partial\alpha_4}+\frac{5}{2}\frac{\partial}{\partial\alpha_5}+\frac{1}{4}\frac{\partial}{\partial\alpha_6})\rho_{\vec{\alpha}}(\vec{x})\mid_{\vec{\alpha}=(0,\alpha_2,-3\alpha_2,-\alpha_2,2\alpha_2-1/12,-1/24)},\\
\rho^{(4)}_{D\ \vec{a}}(\vec{x}) &= (\frac{\partial}{\partial\alpha_2}-\frac{\partial}{\partial\alpha_4}-\frac{1}{2}\frac{\partial}{\partial\alpha_5}+1/4\frac{\partial}{\partial\alpha_6})\rho_{\vec{\alpha}}(\vec{x})\mid_{\vec{\alpha}=(0,\alpha_2,-1/2\alpha_2-1/12,-\alpha_2,-1/2\alpha_2,-1/24)},\\
\rho^{(5)}_{D\ \vec{a}}(\vec{x}) &= (\frac{\partial}{\partial\alpha_3}+\frac{1}{4}\frac{\partial}{\partial\alpha_6})\rho_{\vec{\alpha}}(\vec{x})\mid_{\vec{\alpha}=(0,-2\alpha_3+5/6,\alpha_3,2\alpha_3-11/6,\alpha_3-11/12,-1/24)},\\
\rho^{(6)}_{D\ \vec{a}}(\vec{x}) &= (\frac{\partial}{\partial\alpha_2}+\frac{1}{4}\frac{\partial}{\partial\alpha_6})\rho_{\vec{\alpha}}(\vec{x})\mid_{\vec{\alpha}=(0,\alpha_2,-3\alpha_2-1,-\alpha_2-1,2\alpha_2+11/12,-1/24)}\ . \qquad (13)
\end{aligned}
$$

To find the classical solutions, we use the following relations,

$$
\begin{aligned}
u_1 &= -a_1^2-a_2^2-a_3^2-a_4^2-a_5^2-a_6^2+a_5a_4+a_1a_2+a_6a_3+a_3a_2+a_4a_3,\\
u_2 &= -a_1a_2a_3^2a_4+a_2a_5a_3^2a_4+a_1^2a_3^2a_4+a_1^2a_2a_4^2+a_1^2a_4^2a_5-a_1^2a_4a_5^2-a_1a_2^2a_4^2+a_1a_2^2a_6^2-a_1a_2^2a_5^2-a_1^2a_2a_6^2\\
&+ a_1^2a_2a_5^2+a_1^2a_6^2a_3-a_1^2a_3a_4^2-a_1^2a_3^2a_6+a_2^2a_4^2a_5-a_2^2a_4a_5^2+a_2^2a_3a_5^2-a_1^2a_2a_4a_3+a_1^2a_2a_6a_3-a_1^2a_2a_5a_4\\
&+ a_1a_2a_5^2a_4-a_1a_2a_5a_4^2-a_1a_2^2a_6a_3+a_1a_2^2a_4a_3+a_1a_2^2a_4a_5-a_1a_2a_6^2a_3+a_1a_2a_3a_4^2+a_1a_2a_3^2a_6+a_2a_5^2a_3a_4\\
&- a_2a_5a_3a_4^2-a_2^2a_3a_5a_4-a_5a_6^2a_4^2+a_5^2a_4a_6^2-a_5^2a_3a_6^2+a_5^2a_3^2a_6+a_5a_6^2a_4a_3-a_5^2a_6a_4a_3+a_5a_6a_4^2a_3-a_5a_3^2a_6a_4\\
&- a_2a_5^2a_3^2, \qquad (14)
\end{aligned}
$$

and complicated expressions for u_3, u_4, u_5, u_6 which we do not express here. These relations are obtained through the following relation between the classical part of hyperelliptic curve of ref. [6] and the functions p_1, p_2, q_1 given in (3),

$$
-108x^3W(x) = -108x^3\prod_{i=1}^{27}(x-\tilde{a}_i) = (x^3u_6-q_1)^2-p_1^2p_2 \qquad (15)
$$

where $\tilde{a}_i$ are given in ref. [6]. From the (14), one may construct the classical solutions,

$$
\begin{aligned}
a_1 &= \frac{3}{\sqrt{2}}\rho^{(4)}_{(0,-1/6,0,1/6,1/12,-1/24)}(\vec{x}),\\
a_2 &= \frac{\sqrt{5}}{2}\rho^{(3)}_{(0,1/30,-1/10,-1/30,-1/60,-1/24)}(\vec{x})+\frac{3}{\sqrt{5}}\rho^{(2)}_{\vec{\alpha}}(\vec{x}),
\end{aligned}
$$

$$
\begin{aligned}
a_3 &= \frac{1}{\sqrt{3}}\rho^{(1)}_{\vec{\alpha}}(\vec{x}) + \frac{3}{\sqrt{2}}\rho^{(4)}_{(0,-1/6,0,1/6,1/12,-1/24)}(\vec{x}), \\
a_4 &= \rho^{(2)}_{\vec{\alpha}}(\vec{x}) + \frac{\sqrt{5}}{3}\rho^{(6)}_{(0,-1/4,-1/4,-3/4,5/12,-1/24)}(\vec{x}), \\
a_5 &= \frac{1}{\sqrt{2}}\rho^{(1)}_{\vec{\alpha}}(\vec{x}) + \frac{\sqrt{2}}{3}\rho^{(5)}_{(0,-17/30,7/10,-13/30,-13/60,-1/24)}(\vec{x}) + \\
&\quad \frac{\sqrt{2}}{3}\rho^{(6)}_{(0,-1/4,-1/4,-3/4,5/12,-1/24)}(\vec{x}), \\
a_6 &= \frac{\sqrt{3}}{3}\rho^{(6)}_{(0,-1/4,-1/4,-3/4,5/12,-1/24)}(\vec{x}) + \frac{4}{\sqrt{2}}\rho^{(4)}_{(0,-1/6,0,1/6,1/12,-1/24)}(\vec{x}), \\
a_{D1} &= \frac{2i}{\pi}(\sqrt{3}\rho^{(3)}_{D\ (0,1/30,-1/10,-1/30,-1/60,-1/24)}(\vec{x}) - \frac{1}{3}\sqrt{2}\rho^{(5)}_{D\ (0,-17/30,7/10,-13/30,-13/60,-1/24)}(\vec{x})) + \sum_{i=1}^{6}\epsilon_{1i}\rho^{(i)}_{\vec{\alpha}}(\vec{x}), \\
a_{D2} &= \frac{i}{\pi}(\frac{1}{2}\sqrt{5}\rho^{(2)}_{D\ \vec{\alpha}}(\vec{x}) + \frac{1}{3}\sqrt{5}\rho^{(4)}_{D\ (0,-1/6,0,1/6,1/12,-1/24)}(\vec{x})) + \sum_{i=1}^{6}\epsilon_{2i}\rho^{(i)}_{\vec{\alpha}}(\vec{x}), \\
a_{D3} &= \frac{2i}{\pi}(\sqrt{5}\rho^{(2)}_{D\ \vec{\alpha}}(\vec{x}) + \frac{1}{5}\sqrt{2}\rho^{(4)}_{D\ (0,-1/6,0,1/6,1/12,-1/24)}(\vec{x})) + \sum_{i=1}^{6}\epsilon_{3i}\rho^{(i)}_{\vec{\alpha}}(\vec{x}), \\
a_{D4} &= \frac{4i}{\pi}\sqrt{2}\rho^{(1)}_{D\ \vec{\alpha}}(\vec{x}) + \sum_{i=1}^{6}\epsilon_{4i}\rho^{(i)}_{\vec{\alpha}}(\vec{x}), \\
a_{D5} &= \frac{i}{3\pi}(\frac{1}{2}\rho^{(5)}_{D\ (0,-17/30,7/10,-13/30,-13/60,-1/24)}(\vec{x}) + \sqrt{2}\rho^{(1)}_{D\ \vec{\alpha}}(\vec{x})) + \sum_{i=1}^{6}\epsilon_{5i}\rho^{(i)}_{\vec{\alpha}}(\vec{x}), \\
a_{D6} &= \frac{2i}{\pi}(\frac{\sqrt{3}}{3}\rho^{(4)}_{D\ (0,-1/6,0,1/6,1/12,-1/24)}(\vec{x}) + \frac{\sqrt{2}}{4}\rho^{(6}_{D\,(0,-1/4,-1/4,-3/4,5/12,-1/24)}(\vec{x})) + \sum_{i=1}^{6}\epsilon_{6i}\rho^{(i)}_{\vec{\alpha}}(\vec{x}),
\end{aligned}
\tag{16}
$$

where ϵ_{ij} are constants and are determined by the evaluating of the period integrals. However, for the computation of the instanton correction to the prepotential, explicit form of the $a_{D\,i}$ is not necessary. From these solutions, we get the following identities,

$$
\begin{aligned}
\sum_{i=1}^{6}(\partial_{u_1}a_{D\,i}a_i - a_{D\,i}\partial_{u_1}a_i) &= \frac{12i}{\pi} \\
\sum_{i=1}^{6}(\partial_{u_j}a_{D\,i}a_i - a_{D\,i}\partial_{u_j}a_i) &= 0,\ j\in\{2,\cdots,6\}.
\end{aligned}
\tag{17}
$$

Although the proof of the above equations is difficult, we have explicitly checked (17) up to order w. By integrating the identities (17) over $u_1,\cdots,u_6$, we get the scaling equation,

$$
\sum_{i=1}^{6}a_i\frac{\partial\mathcal{F}}{\partial a_i} - 2\mathcal{F} = \frac{12i}{\pi}u_1. \tag{18}
$$

From the above equation and the following form of the prepotential $F(\vec{a})$ in the semi-classical region,

$$
F(\vec{a}) = \frac{i}{4\pi}\sum_{\alpha\in\Delta_+(E_6)}(\alpha,a)^2\log\frac{(\alpha,a)^2}{w^{1/12}} + 1/2\tau_0\sum_{i=1}^{6}(a_i)^2 + \sum_{n=1}^{\infty}\mathcal{F}_n(\vec{a})w^n, \tag{19}
$$

one can find the n-instanton coefficients $\mathcal{F}_n(\vec{a})$. We get the following form for the one-instanton effect which because of its complexity, only one term of it is given explicitly here,

$$
\begin{aligned}
\mathcal{F}_1(\vec{a}) \quad = \quad & \frac{150}{i\pi}\{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_1-a_6)(a_2-a_3)(a_2-a_4)(a_2-a_5)(a_2-a_6) \\
& ((1-\sqrt{2})(a_1+a_3+a_4)-(1+\sqrt{2})(a_2+a_5+a_6))((1-\sqrt{2})(a_1+a_4+a_5)-(1+\sqrt{2})(a_2+a_3+a_6)) \\
& ((1-\sqrt{2})(a_1+a_5+a_6)-(1+\sqrt{2})(a_2+a_3+a_4))((1-\sqrt{2})(a_1+a_4+a_6)-(1+\sqrt{2})(a_2+a_3+a_5)) \\
& ((1-\sqrt{2})(a_1+a_3+a_5)-(1+\sqrt{2})(a_2+a_4+a_6))((1-\sqrt{2})(a_1+a_3+a_6)-(1+\sqrt{2})(a_2+a_4+a_5)) \\
& ((1+\sqrt{2})(a_1+a_3+a_4)-(1-\sqrt{2})(a_2+a_5+a_6))((1+\sqrt{2})(a_1+a_4+a_5)-(1-\sqrt{2})(a_2+a_3+a_6)) \\
& ((1+\sqrt{2})(a_1+a_4+a_6)-(1-\sqrt{2})(a_2+a_3+a_5))((1+\sqrt{2})(a_1+a_3+a_5)-(1-\sqrt{2})(a_2+a_4+a_6)) \\
& ((1+\sqrt{2})(a_1+a_3+a_6)-(1-\sqrt{2})(a_2+a_4+a_5))\}/\{(a_1-a_2)^2(a_1-a_3)^2(a_1-a_4)^2(a_1-a_5)^2(a_1-a_6)^2 \\
& (a_2-a_3)^2(a_2-a_4)^2(a_2-a_5)^2(a_2-a_6)^2((1-\sqrt{2})(a_1+a_3+a_4)-(1+\sqrt{2})(a_2+a_5+a_6))^2 \\
& ((1-\sqrt{2})(a_1+a_4+a_5)-(1+\sqrt{2})(a_2+a_3+a_6))^2((1-\sqrt{2})(a_1+a_5+a_6)-(1+\sqrt{2})(a_2+a_3+a_4))^2 \\
& ((1-\sqrt{2})(a_1+a_4+a_6)-(1+\sqrt{2})(a_2+a_3+a_5))^2((1-\sqrt{2})(a_1+a_3+a_5)-(1+\sqrt{2})(a_2+a_4+a_6))^2 \\
& ((1-\sqrt{2})(a_1+a_3+a_6)-(1+\sqrt{2})(a_2+a_4+a_5))^2((1+\sqrt{2})(a_1+a_3+a_4)-(1-\sqrt{2})(a_2+a_5+a_6))^2 \\
& ((1+\sqrt{2})(a_1+a_4+a_5)-(1-\sqrt{2})(a_2+a_3+a_6))^2((1+\sqrt{2})(a_1+a_5+a_6)-(1-\sqrt{2})(a_2+a_3+a_4))^2 \\
& ((1+\sqrt{2})(a_1+a_4+a_6)-(1-\sqrt{2})(a_2+a_3+a_5))^2((1+\sqrt{2})(a_1+a_3+a_5)-(1-\sqrt{2})(a_2+a_4+a_6))^2 \\
& ((1+\sqrt{2})(a_1+a_3+a_6)-(1-\sqrt{2})(a_2+a_4+a_5))^2\}+\cdots
\end{aligned} \tag{20}
$$

Redefininhg w as,

$$w_{PV}=\frac{1}{2^1 375}w, \tag{21}$$

we find that one-instanton coefficient (20) is in agreement with the microscopic calculation of [10].

For comparison the strong behaviour of theory based on the hyperelliptic curve, one must construct the corresponding PF equations and solve them, which will appear in a forthcoming paper.

ACKNOWLEDGEMENTS

The author would like to thank M. Alishahiha for helpful discussions.

APPENDIX

Here, we list the functions $A_{\vec{\alpha}}(\vec{m}), \cdots, Y_{\vec{\alpha}}(\vec{m})$ and $Z_{\vec{\alpha}}(\vec{m})$ which appear in the equations (11) and (12),

$$
\begin{aligned}
A_{\vec{\alpha}}(\vec{m}) = \frac{32(\gamma_3+2)(\gamma_3+1)}{3\gamma_6(2\gamma_1-\gamma_2+2\gamma_3-\gamma_5+1)} \quad &, \quad B_{\vec{\alpha}}(\vec{m}) = \frac{3(\gamma_2+1)(\gamma_6+1)}{\gamma_4\gamma_5}, \\
C_{\vec{\alpha}}(\vec{m}) = \frac{4(\gamma_2+1)(\gamma_3+1)}{\gamma_1\gamma_5} \quad &, \quad D_{\vec{\alpha}}(\vec{m}) = \frac{4(\gamma_3+1)(\gamma_4+1)}{3\gamma_1\gamma_6}, \\
E_{\vec{\alpha}}(\vec{m}) = \frac{-(\gamma_1+1)(\gamma_4+1)}{2\gamma_3\gamma_5} \quad &, \quad F_{\vec{\alpha}}(\vec{m}) = \frac{-4(\gamma_3+1)}{\gamma_5}, \\
G_{\vec{\alpha}}(\vec{m}) = \frac{-3(\gamma_6+1)}{\gamma_5} \quad &, \quad H_{\vec{\alpha}}(\vec{m}) = \frac{-2(\gamma_4+1)(\gamma_4+2)}{3\gamma_5\gamma_6}, \\
I_{\vec{\alpha}}(\vec{m}) = F_{\vec{\alpha}}(\vec{m}) \quad &, \quad J_{\vec{\alpha}}(\vec{m}) = G_{\vec{\alpha}}(\vec{m}), \\
K_{\vec{\alpha}}(\vec{m}) = \frac{(\gamma_2+1)(\gamma_2+2)}{18\gamma_6(\gamma_1+\gamma_2)} \quad &, \quad L_{\vec{\alpha}}(\vec{m}) = \frac{-2(\gamma_1+2)(\gamma_4+1)}{3\gamma_6(\gamma_1+\gamma_2)}, \\
M_{\vec{\alpha}}(\vec{m}) = \frac{(\gamma_4+1)(\gamma_4+2)}{6\gamma_6(\gamma_1+\gamma_2)} \quad &, \quad N_{\vec{\alpha}}(\vec{m}) = \frac{4(\gamma_1+1)(\gamma_1+2)}{\gamma_5(-\gamma_1+\gamma_2+\gamma_4}, \\
O_{\vec{\alpha}}(\vec{m}) = \frac{8(\gamma_1-1)(\gamma_3+1)}{\gamma_5(-\gamma_1+\gamma_2+\gamma_4} \quad &, \quad P_{\vec{\alpha}}(\vec{m}) = \frac{-(\gamma_1+1)(\gamma_4+1)}{\gamma_5(-\gamma_1+\gamma_2+\gamma_4}, \\
Q_{\vec{\alpha}}(\vec{m}) = \frac{8\gamma_4(\gamma_3+1)}{\gamma_5(-\gamma_1+\gamma_2+\gamma_4} \quad &, \quad R_{\vec{\alpha}}(\vec{m}) = \frac{2(\gamma_2+2)(\gamma_4+1)}{\gamma_5(-\gamma_1+\gamma_2+\gamma_4}, \\
S_{\vec{\alpha}}(\vec{m}) = \frac{12(\gamma_3+1)}{\gamma_5(-\gamma_1+\gamma_2+\gamma_4} \quad &, \quad T_{\vec{\alpha}}(\vec{m}) = \frac{3(\gamma_6+1)}{\gamma_5(-\gamma_1+\gamma_2+\gamma_4}, \\
U_{\vec{\alpha}}(\vec{m}) = \frac{2(\gamma_2+2)(\gamma_4+1)}{\gamma_5(-\gamma_5+1)} \quad &, \quad V_{\vec{\alpha}}(\vec{m}) = F_{\vec{\alpha}}(\vec{m}), \\
W_{\vec{\alpha}}(\vec{m}) = G_{\vec{\alpha}}(\vec{m}) \quad &, \quad X_{\vec{\alpha}}(\vec{m}) = \frac{(\gamma_4+1)\{2(\gamma_1+\gamma_3)-\gamma_2-\gamma_5-4\}}{8\gamma_1\gamma_3},
\end{aligned}
$$

$$
\begin{aligned}
Y_{\vec{\alpha}}(\vec{m}) &= \{576(\gamma_6+1)(\gamma_6+2)\}/\{4\gamma_1(\gamma_1-1)+20\gamma_1\gamma_2+24\gamma_1\gamma_3+32\gamma_1\gamma_4+36\gamma_1\gamma_5+48\gamma_1\gamma_6+25\gamma_2(\gamma_2-1) \\
&+ 60\gamma_2\gamma_3+80\gamma_2\gamma_4+90\gamma_2\gamma_5+120\gamma_2\gamma_6+36\gamma_3(\gamma_3-1)+96\gamma_3\gamma_4+108\gamma_3\gamma_5+144\gamma_3\gamma_6+64\gamma_4(\gamma_4-1) \\
&+ 144\gamma_4\gamma_5+192\gamma_4\gamma_6+162\gamma_5(\gamma_5-1)+216\gamma_5\gamma_6+144\gamma_6(\gamma_6-1)+15\gamma_2+24\gamma_3+48\gamma_4+63\gamma_5+120\gamma_6+1\}, \\
Z_{\vec{\alpha}}(\vec{m}) &= \frac{-(\gamma_1+1)(\gamma_4+1)}{\gamma_5(-\gamma_1+\gamma_2+\gamma_4)}. \qquad (22)
\end{aligned}
$$

In the above equations, γ_i has been obtained from θ_i given in (6), with the replacement $\varphi_i \rightarrow m_i + \alpha_i$.

References

[1] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19; ibid. **B431** (1994) 484.

[2] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. **B344** (1995) 169;

A. Klemm, W. Lerche, and S. Theisen, Int. J. Mod. Phys. **A11** (1996) 1929;

P. Argyres, A. Faraggi, Phys. Rev. Lett. 73 (1995) 3931;

U. H. Danielsson, B. Sundborg, Phys. Lett. **B358** (1995) 273;

A. Brandhuber, K. Landsteiner, Phys. Lett. **B358** (1995) 73;

P. C. Argyres, A. D. Shapere, Nucl. Phys. **B461** (1996) 437;

U. H. Danielsson, B. Sundborg, Phys. Lett. **B358** (1995) 273.

[3] A. Hanany and Y. Oz, Nucl. Phys. **B452** (1995)283;

A. Hanany, Nucl. Phys. **B466** (1996) 85.

[4] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. **B344** (1995) 169.

[5] U. H. Danielsson, B. Sundborg, Phys. Lett. **B370** (1996) 83;

M. Alishahiha, F. Ardalan, F. Mansouri, Phys. Lett. **B381** (1996) 446;

[6] M. R. Abolhasani, M. Alishahiha, A. M. Ghezelbash, Nucl. Phys. **B480** (1996) 279.

[7] M. Matone, Phys. Lett. **B357** (1995) 342.

H. Ewen, K. Foerger and S. Theisen. Nucl. Phys. **B485** (1997) 63;

J. M. Isidro, A. Mukherjee, J. P. Nunes and H.J. Schnitzer, Nucl. Phys. **B492** (1997) 647; hep-th/8703176; hep-th/9704174.

M. Alishahiha, Phys. Lett. **B398** (1997) 100;hep-th/9703186.

[8] E. Martinec and N. P. Warner, Nucl. Phys. **B459** (1996) 97.

[9] K. Ito, hep-th/9703180.

[10] K. Ito and N. Sasakura, Nucl. Phys. **B484** (1997) 141.

[11] K. Ito and S.-K. Yang, hep-th/9708017.

[12] A. M. Ghezelbash, A. Shafiekhani, M. R. Abolhasani, hep-th/9708073.

[13] W. Lerche and N. P. Warner, hep-th/9608183.